Geometric rep theory: application of geometric constructions to algebra.

$$\underline{\text{Geometry}} \xrightarrow{\text{quantization}} \underline{\text{Algebra}}$$

T^*X $(\mathcal{O}_{T^*X})$ $\rightsquigarrow$ $\mathcal{D}_X$ (sheaf of nc algs)

Modules: Alg. diffeqs.

T^*G/B = flag var. $\rightsquigarrow$ $\mathcal{D}_{G/B}$-mod $\cong$ certain $\mathfrak{g}$-mods

$G = SL_n$: usual flag var.

$\mathfrak{g} = \mathrm{Lie}\, G$.

Aside on G/B : $G = \mathbb{C}$ semisimple (or reductive) alg/Lie group

e.g. $G = SL_n\mathbb{C}$, $Sp_{2n}\mathbb{C}$, $SO_n\mathbb{C}$, E_6, E_7, E_8 etc

$B < G$ Borel subgroup = maximal solvable subgp.

<u>Example:</u> $G = SL_n$, $B = \left\{ \begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}_{\det = 1} \leq SL_n$.

Let $\mathcal{B} := \{$ flags $0 \subsetneq V_1 \subsetneq V_2 \subsetneq \dots \subsetneq V_n = \mathbb{C}^n$, $\dim V_i = i \}$

Note: $GL_n\mathbb{C}$ or $SL_n\mathbb{C}$ acts transitively, isotropy $= B$.

isotropy := stabilizer

stabilizer of $(0 \subseteq \mathbb{C} \subseteq \mathbb{C}^2 \subseteq \dots \mathbb{C}^n) =: \mathcal{F}$

is $\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}$.

$G \times \mathcal{B} \to \mathcal{B}$ transitive induces

$$G/B \xrightarrow{\sim} \mathcal{B}$$

$$gB \longmapsto g \cdot \mathcal{F}.$$

Inside T^*G/B,

e.g. "resolved Kostant Slodowy Slices" $\longmapsto$ finite W-algs.

$\mathbb{C}^{2n}$

$\mathbb{C}^{2n} / \Gamma$, $\Gamma < Sp_{2n}$, finite $\rightsquigarrow (\mathcal{D}_{\mathbb{C}^n})^\Gamma$

$T^*\mathbb{C}^n$
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$\mathbb{C}^{2n}$

or "spherical symplectic refl. algs" Cherednik algs.

Hamiltonian/symplectic reduction, $T^*V /\!/\!/ G \rightsquigarrow$ G-equivariant $\mathcal{D}_V$-mods

These are known as "Higgs branch" varieties/quantization

Includes: · hypertoric (G = torus)

· quiver varieties ($G = \prod GL_{n_i}$

V = sum of $V_{n_i}^* \otimes V_{n_j}$).

Main theorem:

Beilinson-Bernstein localization:

$$\mathcal{D}^{\lambda}_{G/B} \xrightarrow[\Gamma(G/B,\,-)]{\sim}$$ $\mathfrak{g}$-mods with $Z(U\mathfrak{g}) \to \mathbb{C}\cdot \mathrm{Id}$ according to λ

$\lambda = 0$: $\mathcal{D}_{G/B} \longmapsto$ $\mathfrak{g}$-mods where $Z(U\mathfrak{g})$ acts like on triv.

Case $G = SL_n$: usual flag var.

<u>Recall:</u> Defn: A rep V of a Lie alg $\mathfrak{g}$ is a homomorphism $\rho: \mathfrak{g} \longrightarrow \mathfrak{gl}(V) = \mathrm{End}\, V$

$$[x,y] \longmapsto \rho(x)\rho(y) - \rho(y)\rho(x)$$

<u>Defn</u>: A rep V of an assoc alg A is a homom $A \longrightarrow \mathrm{End}\, V$.

Class of reps forms an <u>abelian cat</u> in both cases.

<u>Defn</u> $\mathfrak{g}$ = lie alg

$$U\mathfrak{g} := \underset{\substack{\shortparallel \\ \mathfrak{g} \longrightarrow \bigoplus_{m \geq 0} \mathfrak{g}^{\otimes m}}}{T\mathfrak{g}} \Big/ \left(x \otimes y - y \otimes x - [x,y]\right)$$

Have functor U: Lie algs $\longrightarrow$ assoc

$$\xleftarrow{\text{Forgetful}}$$

Induces: F: Rep $\mathfrak{g} \longrightarrow$ Rep $U\mathfrak{g}$

$$(V, \rho) \longmapsto (V, \tilde{\rho}),$$

$$\begin{array}{ccc} \tilde{\rho} : U\mathfrak{g} & \longrightarrow & \operatorname{End} V \\ \uparrow \scriptstyle{PBW} & & \| \\ \rho : \mathfrak{g} & \longrightarrow & \mathfrak{gl} V \end{array}$$

uniquely extends

<u>Exercise</u>: This F is an equivalence, inverse: $\rho \longmapsto \rho|_{\mathfrak{g}}$.

Analogue for finite groups:

Recall $\operatorname{Rep} G \longrightarrow \operatorname{Rep} \mathbb{C}[G]$ is an equiv.

$\Rightarrow$ reps of fin groups $\hookrightarrow$ reps of algs.

Rep theory of semisimple lie algs:

Humphreys, Fulton + Harris, ...

Ex: $\mathfrak{sl}_2 = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \right\} \supseteq \mathfrak{b} = \left\{ \begin{pmatrix} a & b \\ 0 & -a \end{pmatrix} \right\}$

$tr = 0$ "Borel"

$$\mathfrak{b} \supseteq \mathfrak{h} = \left\{ \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} \right\}.$$

"Cartan"

__Thm__ $\exists!$ irrep of sl_2 of each dim $n \geq 1$,

Characterized by: $V_n \Big|_{\mathfrak{h}} \cong \mathbb{C}_{1-n} \overset{\mathbb{C}_{3-n}}{\oplus} \cdots \oplus \mathbb{C}_{n-1}$

$\mathbb{C}_k$ = 1-dim rep, $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ act by $k \cdot Id = (k)$.

__Geometric:__ $V_n = \mathbb{C}[x,y]_{n-1} \leftarrow$ degree, $gl_2 \supseteq sl_2$ acts as on $\mathbb{C}\cdot x \oplus \mathbb{C}\cdot y \cong \mathbb{C}^2$

Span$(x^{n-1}, x^{n-2}y, \ldots)$

Derivation

$\xi \in \mathfrak{sl}_2$, $\xi(x^m y^n) = m\,\xi(x)\,x^{m-1} y^n + n\,\xi(y)\,x^m y^{n-1}$.

i.e. $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ acts by $a x \partial_x + b x \partial_y + c y \partial_x + d y \partial_y$.

<u>Relation to $\mathbb{P}^1$</u>: $\mathbb{C}[x,y]_n = \Gamma(\mathbb{P}^1, \mathcal{O}(n))$.

GL_2 acts on $\mathbb{P}^1$, ($GL_2 \to PGL_2 \cong \operatorname{Aut}\mathbb{P}^1$).

$\rightsquigarrow$ action on $\Gamma(\mathbb{P}^1, \mathcal{O}(n))$.

General case: $\mathfrak{g}$ = semisimple Lie alg / $\mathbb{C}$

(a direct sum of simple (nonabelian) Lie algs.)

$\mathfrak{b}$ = maximal $\underbrace{\text{solvable}}_{\exists m,\ \mathfrak{b}^{(m)} = 0,}$ subalgebra

$$\mathfrak{b}^{(k+1)} = [\mathfrak{b}^{(k)}, \mathfrak{b}^{(k)}].$$

$\mathfrak{n} := [\mathfrak{b}, \mathfrak{b}]$, $\mathfrak{h} \subseteq \mathfrak{b}$ "Cartan subalg", any abelian subalg, $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$.

Ex $\mathfrak{g} = \underset{tr=0}{\mathfrak{sl}_n} \supseteq \left\{ \underset{tr=0}{\begin{pmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{pmatrix}} \right\} = \mathfrak{b} \supseteq \mathfrak{h} = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$

$\mathfrak{n} = \left\{ \begin{pmatrix} 0 & & * \\ & \ddots & \\ 0 & & 0 \end{pmatrix} \right\}.$

Thm 1) $\overset{\text{abelian}}{\mathfrak{h}}$ acts semisimply on any f.d. rep,

(V, ρ) : $(V, \rho|_{\mathfrak{h}}) \cong \bigoplus_{\lambda \in I_V} (\mathbb{C}_\lambda^{r_\lambda}, \rho_\lambda)$

$I_V \subseteq \mathfrak{h}^*$, $\mathbb{C}_\lambda$ = one-dim rep, $\rho_\lambda(h) = (\lambda(h))$.

2) If V irr, then $\exists!\ \lambda \in I_V$ "highest wt" s.t. V generated over $\mathfrak{b}$ by v_λ, $\rho(x)(v_\lambda) = \lambda(x)\, v_\lambda$, $x \in \mathfrak{h}$, then $r_\lambda = 1$.

$\Rightarrow$ $\{$irr fd reps of $\mathfrak{g}\}/\sim \;\hookrightarrow\; \mathfrak{h}^*$

Image = "dominant integral weights"

3) All fd reps are semisimple ($\oplus$ of irreps).

Note: Given (dominant integral) $\lambda \in \mathfrak{h}^*$, there is a constr. of (f.d.) irrep of $\mathfrak{g}$, h.w. λ.

Geometry: G = (almost) semisimple alg/Lie gp (ie Lie G is semisimple)

$B < G$ Borel (max (solv. $\Leftrightarrow$ Lie B solvable))

Flag var $:= G/B$

Let $T < B$ be maximal torus $(\cong (\mathbb{C}^\times)^m)$

$\Rightarrow$ $\mathfrak{h} := \mathrm{Lie}\, T$, $\mathfrak{b} := \mathrm{Lie}\, B$ are Cartan, Borel.

Let $\lambda \in \underset{\substack{\text{group} \\ \text{lattice}}}{\mathrm{Hom}}(T, \mathbb{C}^\times) \subseteq \mathfrak{h}^*$

Define $\mathcal{O}(-\lambda) := G \times_B \mathbb{C}_\lambda$, $\mathbb{C}_\lambda$ rep of $B \supseteq T$

$(G \times \mathbb{C}_\lambda)/B$

$b \cdot v = \exp(\lambda(h))v$

$b = \exp(h+n)$

$h \in \mathfrak{h}$, $n \in \mathfrak{n}$. (As rep of $\mathfrak{b}$, $\mathfrak{n}$ acts by 0

$\mathfrak{h} \hookrightarrow \mathfrak{b} \twoheadrightarrow \mathfrak{b}/\mathfrak{n}$, $\mathfrak{h} \xrightarrow{\cong} \mathfrak{b}/\mathfrak{n}$

Remember $h \cdot v = \lambda(h) v$, $h \in \mathfrak{h}$.)

Here $G \times \mathbb{C}_\lambda$ is line bundle on G,

$$B \circlearrowright G, \quad B \circlearrowright \mathbb{C}_\lambda, \quad B \twoheadrightarrow T$$

$$G \times_B \mathbb{C}_\lambda := (G \times \mathbb{C}_\lambda)/B = \text{line bdle on } G/B.$$

Thm (Borel–Weil): $\Gamma(G/B, \mathcal{O}(\lambda)) = 0$ or (λ NOT dominant)

h.w. irrep of h.w. λ (λ dominant)

eg. $G = SL_2$, $G/B = \mathbb{P}^1$: $\mathcal{O}(m)$,

m dominant $\Leftrightarrow m \geq 0$.

$\Gamma(\mathbb{P}^1, \mathcal{O}(-k)) = 0$, $k \geq 1$

Note: G/B is projective.

<u>Remark:</u> Borel–Weil–Bott extends to

$$H^i(G/B, \mathcal{O}(\lambda)).$$

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geometry $\rightsquigarrow$ rep thy of $\mathfrak{g}$ s.s.

Only: finite-dim reps.

<u>Deeper:</u> observe: $\mathfrak{g} \subseteq \mathrm{Vect}(G/B) := \Gamma(G/B, T_{G/B})$

$G \curvearrowright G/B$

If we have sheaf on G/B, $\mathcal{F}$,
and action of $T_{G/B}$ on $\mathcal{F}$

$\Rightarrow \Gamma(\mathcal{F}) \curvearrowleft \mathfrak{g}$.

$$\mathcal{D}^{\lambda}_{G/B} := \{\text{diff ops } \mathcal{O}(\lambda) \to \mathcal{O}(\lambda)\}$$

$\lambda = 0$: $\mathcal{D}_{G/B}$ (gen by $\mathcal{O}, T$)

(makes sense $\lambda \in \mathfrak{h}^* \cong \mathrm{Hom}(T, \mathbb{G}^\times)$)

Thm Beilinson–Bernstein:

1) $\Gamma(G/B, \mathcal{D}^{\lambda}_{G/B}) \cong U\mathfrak{g} / (\ker \sigma^*(\lambda))$

(Harish-Chandra Isom: $\sigma : Z(U\mathfrak{g}) \xrightarrow{\sim} \mathbb{C}[\mathfrak{h}^*]^W$

$\boxed{W = \text{weyl group}}$

2) $\Gamma : \mathcal{D}^{\lambda}_{G/B}\text{-mod} \xrightarrow[\sim]{\Gamma(G/B, -)} \mathfrak{g}\text{-mods with}$

$Z(U\mathfrak{g})$ acting by $\sigma^*(\lambda)$

Under thm, $(O \subset N) \longmapsto V_\lambda$

h.w. λ irr.

Symplectic geom:

$$\left(T^*G/B \xrightarrow[\text{symplectic resol}]{} \underset{\mathfrak{g}}{\overset{\text{cone.}}{\mathrm{Nil}\,\mathfrak{g}}} \subset \mathfrak{g} = \text{ad-nilp elements.} \right)$$